A COMPACT UNIVERSAL DIFFERENTIABILITY SET WITH HAUSDORFF DIMENSION ONE

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Abstract. We give a short proof that any non-zero Euclidean space has a compact subset of Hausdorff dimension one that contains a differentiability point of every real-valued Lipschitz function defined on the space.

1. Introduction

1.1. It is well known that if $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then f is differentiable almost everywhere. Zahorski, [10], gives a full characterisation of the possible sets of points of non-differentiability of a real-valued Lipschitz function defined on $\mathbb{R}$. In particular, it follows that for any Lebesgue null set $E \subseteq \mathbb{R}$ there exists a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ that is non-differentiable at every point of E.

Turning to higher dimensions, we may still conclude that real-valued functions defined on a finite dimensional Euclidean space are differentiable almost everywhere; this is Rademacher's theorem. However the converse implication no longer holds; in any Euclidean space of dimension at least two, there are sets of measure zero on which every real-valued Lipschitz function, defined on the space, is somewhere differentiable. Examples of sets satisfying this property — which we hitherto refer to as the universal differentiability property — were constructed by Preiss in [9] and the authors of the present paper in [3].

It is proved in [9] that if $E \subseteq \mathbb{R}^d$ is any G_δ set, i.e. an intersection of a countable family of open sets, such that E contains every line segment that passes through any two points of some dense subset $S \subseteq \mathbb{R}^d$, then E has the universal differentiability property. One can check that not only can such a set E be taken to be null in $\mathbb{R}^d$ for $d \geq 2$ but that such a set may also be chosen to have Hausdorff dimension one; see Lemma 1.3. However it is also clear that the closure of such a set E is the whole space $\mathbb{R}^d$.

In [3] we show, on the other hand, that it is possible to find a compact and null subset $E \subseteq \mathbb{R}^d$ with the universal property, for $d \geq 2$. An example for $d = 2$ is given

by a generalisation of the Menger-Sierpinski carpet; see also [4]. More precisely we choose odd integers $N_i > 1$ such that $N_i \to \infty$ and $\sum N_i^{-2} = \infty$. At the first step, we divide the unit square $[0,1]^2$ into N_1^2 equal squares and remove the central square in the division. Then on each subsequent step we divide the remaining squares into N_i^2 equal squares and remove the central square. The example is given by the set that remains. For $d > 2$ we take the Cartesian product of this two-dimensional set with $\mathbb{R}^{d-2}$. See [3] for more details.

The drawback of the example in [3] is, however, that its Hausdorff dimension is equal to d, the dimension of the underlying space.

In the present paper we construct a compact subset of $\mathbb{R}^d$ with the universal differentiability property such that its Hausdorff dimension is equal to one, thus making the set small both in terms of its closure and in terms of its Hausdorff dimension.

For Lipschitz mappings to spaces of dimension larger than one there are fewer positive results. For $d \geq 3$, it is proved in [8] that there exists a Lebesgue null set E in $\mathbb{R}^d$ such that every Lipschitz mapping from $\mathbb{R}^d$ to $\mathbb{R}^{d-1}$ has a point of ε-differentiability in that set for all $\varepsilon > 0$; see the subsequent section for a definition. In fact one may take E to be the union of all "rational hyperplanes" in $\mathbb{R}^d$, so that the Hausdorff dimension of E is equal to $d-1$. However ε-differentiability is weaker than differentiability.

1.2. Recall for a pair of real Banach spaces X, Y a function $f : X \to Y$ is called Lipschitz if there is a constant $L \geq 0$ such that

$$\|f(x') - f(x)\|_Y \leq L\|x' - x\|_X$$

for any $x, x' \in X$. The smallest such L is denoted as $\mathrm{Lip}(f)$.

We say that the function $f : X \to Y$ has a directional derivative at $x \in X$ in the direction $e \in X$ if the limit

$$\lim_{t\to 0} \frac{f(x+te) - f(x)}{t} \tag{1.1}$$

exists. We then denote the limit (1.1) as $f'(x,e)$. We say f is Gâteaux differentiable at $x \in X$ if $f'(x,e)$ exists for every $e \in X$ and $T(e) := f'(x,e)$ defines a bounded linear operator $T : X \to Y$.

If f is Gâteaux differentiable at x and the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - T(h)}{\|h\|} \tag{1.2}$$

is equal to 0 — or equivalently, the convergence in (1.1) is uniform for e in the unit sphere of X — then we say that f is Fréchet differentiable at x and denote the operator T as $f'(x)$.

The condition (1.2) can be rewritten as follows. We require that there exists a bounded linear operator $f'(x)\colon X \to Y$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $h \in X$ with $\|h\| < \delta$ we have

$$\|f(x+h) - f(x) - f'(x)(h)\| \leq \varepsilon \|h\|.$$

If, on the other hand, we only know the existence of such an operator for some fixed ε, we say that f is ε-Fréchet differentiable at x.

We refer the reader to [5, 6] where the notion of ε-Fréchet differentiability is studied in relation to Lipschitz mappings, with the emphasis on the infinite dimensional case. In general, Fréchet differentiability is a strictly stronger property than Gâteaux differentiability. However the two notions coincide for Lipschitz functions defined on a finite dimensional space; see [2]. Hence, in this case, we may simply refer to differentiability, without any ambiguity.

The Hausdorff dimension of a set E, a subset of a metric space, is defined in the following way. For $r, \delta > 0$, let

$$\mathcal{H}^r_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} [\mathrm{diam}(S_i)]^r \text{ over all } S_i \text{ with } E \subseteq \bigcup_{i=1}^{\infty} S_i \text{ and } \mathrm{diam}(S_i) \leq \delta \right\},$$

and

$$\mathcal{H}^r(E) = \lim_{\delta \to 0^+} \mathcal{H}^r_\delta(E); \tag{1.3}$$

as $\mathcal{H}^r_\delta(E)$ is a decreasing function of $\delta > 0$, this limit exists in $[0, \infty]$. The number

$$\dim_{\mathcal{H}}(E) := \inf\{r > 0 \text{ with } \mathcal{H}^r(E) = 0\} \tag{1.4}$$

is called the Hausdorff dimension of E. It is easy to see that the Hausdorff dimension is a monotone set function with respect to inclusion and if $f\colon X \to Y$ is a Lipschitz function then the Hausdorff dimension of $f(E)$ does not exceed the Hausdorff dimension of E, for every $E \subseteq X$. See [7] for a discussion of the properties of Hausdorff dimension.

From this point, we shall work in $\mathbb{R}^d$ where $d \geq 1$. We fix some notation for the rest of the paper. We let $\|\cdot\|$ be the Euclidean distance on $\mathbb{R}^d$ and denote an open ball centered at $a \in \mathbb{R}^d$ of radius r by $B_r(a)$ and a closed ball by $\overline{B}_r(a)$. Further, for

any $S \subseteq \mathbb{R}^d$ and $r \geq 0$ we let

$$B_r(S) := \{x \in \mathbb{R}^d \text{ such that } \inf_{y \in S} \|x - y\| < r\} = \bigcup_{y \in S} B_r(y),$$
$$\overline{B}_r(S) := \{x \in \mathbb{R}^d \text{ such that } \inf_{y \in S} \|x - y\| \leq r\}$$

denote the open and closed r-neighbourhoods of S respectively.

We require the following simple observation.

Lemma 1.3. *If $L \subseteq \mathbb{R}^d$ is a countable union of line segments then there exists a G_δ set $O \subseteq \mathbb{R}^d$ with $L \subseteq O$ such that the Hausdorff dimension of O is equal to one.*

Proof. It clearly suffices to show that the Hausdorff dimension of O can be taken to be less than or equal to one.

Note that if I is a line segment of length at most 1 in a Banach space Y, $r, \delta > 0$ and $k \geq 4/\delta$ is a positive integer then

$$\mathcal{H}^r_\delta(B_{1/k}(I)) \leq k \cdot \left(\frac{4}{k}\right)^r = 4^r \cdot k^{-(r-1)}, \tag{1.5}$$

as we may cover $B_{1/k}(I)$ with k open balls whose radii are equal to $2/k$, i.e. with diameters $4/k \leq \delta$.

Now let $L \subseteq \mathbb{R}^d$ be a countable union of line segments. One may write $L = \bigcup_{m \geq 1} L_m$, where each L_m is a line segment of length at most 1. Let

$$O_n = \bigcup_{m=1}^{\infty} B_{1/2^{m+n}}(L_m) \qquad \text{and} \qquad O = \bigcap_{n=1}^{\infty} O_n.$$

Note that O is a G_δ subset of $\mathbb{R}^d$, containing L. To verify that the Hausdorff dimension of O is no greater than one, it suffices, by (1.3) and (1.4), to show that $\mathcal{H}^r_\delta(O) = 0$ for every $\delta > 0$ and $r > 1$. If $2^{n+1} \geq 4/\delta$, then

$$\begin{aligned}\mathcal{H}^r_\delta(O) \leq \mathcal{H}^r_\delta(O_n) \leq \sum_{m=1}^{\infty} \mathcal{H}^r_\delta(B_{1/2^{m+n}}(L_m)) \leq \sum_{m=1}^{\infty} 4^r \cdot 2^{-(m+n)(r-1)} \\ = 4^r \cdot \frac{2^{-(n+1)(r-1)}}{1 - 2^{-(r-1)}},\end{aligned}$$

using the countable subadditivity of $\mathcal{H}^r_\delta$, (1.5) and $r > 1$. Letting $n \to \infty$ we obtain $\mathcal{H}^r_\delta(O) = 0$ as required. $\square$

1.4. We have already mentioned that by [9] any G_δ set O that contains every line segment passing through two points of a dense subset $R \subseteq \mathbb{R}^d$ has the universal differentiability property and that if R is countable, O may be taken to have Hausdorff dimension one by Lemma 1.3. Our strategy then is to construct a closed and bounded subset of such a set O that still has the universal differentiability property; this will give our example of a compact universal differentiability set with Hausdorff dimension one.

The basic idea of the construction is as follows. We write $O = \bigcap_{k\geq 1} O_k$ where O_k are open subsets of $\mathbb{R}^d$ with $O_{k+1} \subseteq O_k$ for each $k \geq 1$. Then for each $k \geq 1$ we construct a family of closed subsets $M_k(\lambda)_{\lambda\in[0,1]}$ of O_k with the property that $M_k(\lambda) \subseteq M_k(\lambda')$ for $\lambda \leq \lambda'$. Taking the intersection $T_\lambda := \bigcap_{k\geq 1} M_k(\lambda)$ we note that each T_λ is a closed subset of O and that $T_\lambda \subseteq T_{\lambda'}$ for $\lambda \leq \lambda'$. We then prove, using the details of the construction, that the family $(T_\lambda)_{\lambda\in[0,1]}$ contains, in a certain sense, a large amount of line segments connecting two points in the dense set R.

Next, by quoting Theorem 2.7, which is Theorem 3.1 in [3], we show that given a Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$ we can find a point $x \in T_\lambda$ for some $\lambda < 1$ and a direction $e \in S^{d-1}$, the unit sphere of $\mathbb{R}^d$, such that the directional derivative $f'(x,e)$ is almost locally maximal: if $\varepsilon > 0$ and $x' \in T_{\lambda'}$ is close to x, with λ' sufficiently close to λ, and $e' \in S^{d-1}$ is a direction such that (x', e') satisfies certain additional constraints, then $f'(x', e') < f'(x, e) + \varepsilon$.

Finally we then prove f is differentiable at x with derivative

$$f'(u) = f'(x, e)\langle u, e\rangle$$

using Lemma 2.8, which we quote from [3, Lemma 4.3]. This last step makes essential use of the fact that the family $(T_\lambda)_{\lambda\in[0,1]}$ contains sufficiently many line segments.

We finish this section by noting that the Hausdorff dimension of one is optimal.

Lemma 1.5. *If $d \geq 1$ and $E \subseteq \mathbb{R}^d$ is a universal differentiability set, then the Hausdorff dimension of E is at least one.*

Proof. Assume E has Hausdorff dimension strictly less than 1. Let v be any unit vector in $\mathbb{R}^d$ and set

$$\varphi(x) = \langle x, v\rangle.$$

Since $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ is Lipschitz, we conclude

$$\dim_{\mathcal{H}}(\varphi(E)) \leq \dim_{\mathcal{H}}(E) < 1;$$

in particular $\varphi(E) \subseteq \mathbb{R}$ has Lebesgue measure 0. Hence there exists a Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ that is non-differentiable at every $x \in \varphi(E)$. Then $f := g \circ \varphi$ defines a Lipschitz function from $\mathbb{R}^d$ to $\mathbb{R}$ that is not differentiable at every $x \in E$, as the directional derivative $f'(x, v)$ does not exist for $x \in E$. $\square$

2. Construction

Let R be a countable dense subset of $B_1(0)$ and for each $\varepsilon > 0$ let $R(\varepsilon)$ be a finite subset of R such that for every $x \in B_1(0)$ there exists $r \in R(\varepsilon)$ with $\|r - x\| < \varepsilon$.

By Lemma 1.3 we may pick a G_δ set $O \subseteq B_1(0)$ of Hausdorff dimension one such that $[r, s] \subseteq O$ for every $r, s \in R$. We write $O = \bigcap_{k=1}^{\infty} O_k$ where O_k are open subsets of $\mathbb{R}^d$ with $O_{k+1} \subseteq O_k$ for each $k \geq 1$.

Definition 2.1. *For $k \geq 0$ we define compact sets $R_k \subseteq O$ and $w_k > 0$ as follows. Let $w_0 > 0$ and R_0 be any compact subset of R; for example $w_0 = 1$ and $R_0 = \emptyset$. For each $k \geq 1$ we let*

- $R_k = \bigcup\{[r, s]$ *where* $r, s \in R(w_{k-1}/k)\} \cup R_{k-1}$,
- $w_k \in (0, w_{k-1}/2)$ *be such that* $\overline{B}_{w_k}(R_k) \subseteq O_k$.

Since $[r, s] \subseteq O$ for every $r, s \in R(w_{k-1}/k) \subseteq R$ and $R(w_{k-1}/k)$ is finite, the set R_k defined above is compact and is a subset of O. Then we may pick $w_k > 0$ as given above because $R_k \subseteq O \subseteq O_k$, R_k is compact and O_k is open. Note that (w_n) is a decreasing sequence that tends to zero.

Definition 2.2. *If $k \geq 1$ and $\lambda \in [0, 1]$ we set*

$$M_k(\lambda) = \bigcup_{k \leq n \leq (1+\lambda)k} \overline{B}_{\lambda w_n}(R_n). \tag{2.1}$$

We note that for each $k \geq 1$ and $\lambda \in [0, 1]$, the set $M_k(\lambda)$ is a finite union of closed sets, so closed. As $\overline{B}_{\lambda w_n}(R_n) \subseteq O_n$ for any $n \geq 1$ we have, for $k \geq 1$,

$$M_k(\lambda) \subseteq \bigcup_{k \leq n \leq (1+\lambda)k} O_n = O_k \tag{2.2}$$

as $O_{n+1} \subseteq O_n$ for all $n \geq 1$.

Definition 2.3. *Given $\lambda \in [0, 1]$ we set*

$$T_\lambda = \bigcap_{k=1}^{\infty} M_k(\lambda). \tag{2.3}$$

Note from (2.2) that $T_\lambda \subseteq O_k$ for every $k \geq 1$ so that $T_\lambda \subseteq O$ for every $\lambda \in [0,1]$, and the set O is bounded and has Hausdorff dimension one. Further, as T_λ is an intersection of closed sets, it is closed. We conclude that for every $\lambda \in [0,1]$, the set T_λ is a compact subset of $\mathbb{R}^d$ of Hausdorff dimension at most one. Finally we note that if $\lambda_1 \leq \lambda_2$ then we have $T_{\lambda_1} \subseteq T_{\lambda_2}$ and if $\lambda = 0$, Definition 2.2 implies $M_k(0) = R_k$, so that by Definition 2.3 we have $T_0 = \bigcap_{k=1}^{\infty} M_k(0) = R_1$.

Lemma 2.4. *Suppose that* $0 \leq \lambda < \lambda + \psi \leq 1$ *and* $x \in M_k(\lambda)$ *where* $k \geq 1$*. If*

$$0 < \Delta \leq \psi w_n$$

for all $n \leq (1+\lambda)k$ *then we have*

$$B_\Delta(x) \subseteq M_k(\lambda + \psi).$$

Proof. Using Definition 2.2 we may find n such that $k \leq n \leq (1+\lambda)k$ and $x \in \overline{B}_{\lambda w_n}(R_n)$. Noting that $\Delta \leq \psi w_n$ we have

$$B_\Delta(x) \subseteq \overline{B}_{\lambda w_n + \Delta}(R_n) \subseteq \overline{B}_{(\lambda+\psi)w_n}(R_n) \subseteq M_k(\lambda + \psi)$$

using Definition 2.2 once more and $k \leq n \leq (1 + \lambda + \psi)k$. □

Lemma 2.5. *Suppose that* $0 \leq \lambda < \lambda + \psi \leq 1$*. If* $\eta \in (0,1)$*,* $\Delta > 0$ *and* $k \geq 1/(\psi\eta)$ *satisfy* $\Delta > \psi w_n$ *for some* $n \leq (1+\lambda)k$ *then there exists* $\alpha \in (0, \eta\Delta)$ *such that* $[r,s] \subseteq M_l(\lambda + \psi)$ *for every* $r, s \in R(\alpha)$ *and* $l \geq k$*.*

Proof. As the sequence w_n is decreasing we may assume $k \leq n \leq (1+\lambda)k$ so that using $k \geq 1/(\psi\eta)$, we get

$$\alpha := \frac{w_n}{n+1} < \frac{\Delta/\psi}{1/(\psi\eta)} = \eta\Delta.$$

Let $l \geq k$ and choose any $r, s \in R(\alpha)$ so that $[r,s] \subseteq R_{n+1}$ by Definition 2.1. Note that as $n \leq (1+\lambda)k$ and $\psi k \geq 1/\eta \geq 1$,

$$(1 + \lambda + \psi)l \geq (1+\lambda)k + \psi k \geq n + 1.$$

Thus we may pick $m \geq n+1$ with $l \leq m \leq (1+\lambda+\psi)l$. Then $[r,s] \subseteq R_{n+1} \subseteq R_m$. Hence $[r,s] \subseteq M_l(\lambda+\psi)$ by Definition 2.2. □

Lemma 2.6. *For each* $\eta, \psi > 0$ *there exists*

$$\Delta_0 = \Delta_0(\eta, \psi) > 0$$

such that if $\Delta \in (0, \Delta_0)$*,* $0 \leq \lambda < \lambda + \psi \leq 1$ *and* $x \in T_\lambda$ *there exists* $\alpha \in (0, \eta\Delta)$ *such that for every* $r, s \in R(\alpha) \cap B_\Delta(x)$ *we have* $[r,s] \subseteq T_{\lambda+\psi}$*.*

Proof. Pick $\Delta_0 > 0$ with $\Delta_0 < \psi w_n$ for every $n \leq 2/(\psi\eta)$. Now suppose that $\Delta \in (0, \Delta_0)$. Pick a minimal $k \geq 1$ such that $\Delta > \psi w_n$ for some $n \leq (1+\lambda)k$. Note that $(1+\lambda)k > 2/(\psi\eta)$ so that $k > 1/(\psi\eta)$. Thus by Lemma 2.5 we can find $\alpha \in (0, \eta\Delta)$ with $[r,s] \subseteq M_l(\lambda+\psi)$ for every $r, s \in R(\alpha)$ and $l \geq k$. But for $l < k$ if $r, s \in B_\Delta(x)$ then by the minimality of k we have $\Delta \leq \psi w_n$ for every $n \leq (1+\lambda)l$ so that by Lemma 2.4,

$$[r,s] \subseteq B_\Delta(x) \subseteq M_l(\lambda+\psi).$$

Hence for every $r, s \in R(\alpha) \cap B_\Delta(x)$ we have $[r,s] \subseteq M_l(\lambda+\psi)$ for any $l \geq 1$, so that $[r,s] \subseteq T_{\lambda+\psi}$. □

We now let the Hilbert space H equal $\mathbb{R}^d$ and write $S(H)$ for the unit sphere of H. Note that $(\mathfrak{S}, \preceq) := ([0,1], \leq)$ is a dense, chain complete poset: for any $\lambda_1 < \lambda_2$ there exists $\lambda \in (\lambda_1, \lambda_2)$ and every non-empty chain in $([0,1], \leq)$ has a supremum.

We quote [3, Theorem 3.1] as Theorem 2.7. The assumptions for Theorem 2.7 are as follows: H is a real Hilbert space, $(\mathfrak{S}, \preceq)$ is a dense chain complete poset and $(T_\lambda)_{\lambda\in\mathfrak{S}}$ is a collection of closed subsets of H such that $T_\lambda \subseteq T_{\lambda'}$ whenever $\lambda \preceq \lambda'$.

We also use the following notation. For a Lipschitz function $h\colon H \to \mathbb{R}$ we write D^h for the set of all pairs $(x,e) \in H \times S(H)$ such that the directional derivative $h'(x,e)$ exists and, for each $\lambda \in \mathfrak{S}$, we let D^h_λ be the set of all $(x,e) \in D^h$ such that $x \in T_\lambda$. If, in addition, $h\colon H \to \mathbb{R}$ is linear then we write $\|h\|$ for the operator norm of h.

Theorem 2.7. *Suppose $f_0\colon H \to \mathbb{R}$ is a Lipschitz function, $\lambda_0 \in \mathfrak{S}$, $(x_0, e_0) \in D^{f_0}_{\lambda_0}$, $\delta_0, \mu, K > 0$ and $\lambda_1 \in \mathfrak{S}$ with $\lambda_0 \prec \lambda_1$. Then there exists a Lipschitz function $f\colon H \to \mathbb{R}$ such that $f - f_0$ is linear with norm not greater than μ and a pair $(x,e) \in D^f_\lambda$, where $\|x - x_0\| \leq \delta_0$ and $\lambda \in (\lambda_0, \lambda_1)$, such that the directional derivative $f'(x,e) > 0$ is almost locally maximal in the following sense. For any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ and $\lambda_\varepsilon \in (\lambda, \lambda_1)$ such that whenever $(x', e') \in D^f_{\lambda_\varepsilon}$ satisfies*

(i) $\|x' - x\| \leq \delta_\varepsilon$, $f'(x', e') \geq f'(x,e)$ *and*
(ii) *for any* $t \in \mathbb{R}$

$$|(f(x'+te) - f(x')) - (f(x+te) - f(x))| \leq K\sqrt{f'(x',e') - f'(x,e)}\,|t|, \tag{2.4}$$

then we have $f'(x', e') < f'(x,e) + \varepsilon$.

We now quote [3, Lemma 4.3].

Lemma 2.8 (Differentiability Lemma)**.** *Let H be a real Hilbert space, $f : H \to \mathbb{R}$ be a Lipschitz function and $(x, e) \in H \times S(H)$ be such that the directional derivative $f'(x, e)$ exists and is non-negative. Suppose that there is a family of sets $\{F_\varepsilon \subseteq H \mid \varepsilon > 0\}$ such that*

(1) *whenever $\varepsilon, \eta > 0$ there exists $\delta_* = \delta_*(\varepsilon, \eta) > 0$ such that for any $\delta \in (0, \delta_*)$ and u_1, u_2, u_3 in the closed unit ball of H, one can find u'_1, u'_2, u'_3 with $\|u'_m - u_m\| \leq \eta$ and*
$$[x + \delta u'_1, x + \delta u'_3] \cup [x + \delta u'_3, x + \delta u'_2] \subseteq F_\varepsilon,$$

(2) *whenever $(x', e') \in F_\varepsilon \times S(H)$ is such that the directional derivative $f'(x', e')$ exists, $f'(x', e') \geq f'(x, e)$ and*
$$\begin{aligned} &|(f(x' + te) - f(x')) - (f(x + te) - f(x))| \\ &\leq 25\sqrt{(f'(x', e') - f'(x, e))\mathrm{Lip}(f)}|t| \end{aligned} \tag{2.5}$$
for every $t \in \mathbb{R}$ then
$$f'(x', e') < f'(x, e) + \varepsilon. \tag{2.6}$$

Then f is Fréchet differentiable at x and its derivative $f'(x)$ is given by the formula
$$f'(x)(h) = f'(x, e)\langle h, e\rangle \tag{2.7}$$
for $h \in H$.

We now apply these results to our construction to obtain the following.

Theorem 2.9. *There exists a compact subset $S \subseteq \mathbb{R}^d$ of Hausdorff dimension one with the universal differentiability property; moreover if $g \colon \mathbb{R}^d \to \mathbb{R}$ is Lipschitz, the set of points $x \in S$ such that f is Fréchet differentiable at x is a dense subset of S.*

Proof. We let
$$S = \overline{\bigcup_{q<1} T_q}.$$
We first note $S \supseteq T_0 = \bigcap_{k=1}^{\infty} M_k(0) = R_1 \neq \emptyset$ so that S is not empty. Also as T_1 is closed and $S \subseteq T_1 \subseteq O \subseteq B_1(0)$, we conclude S is a compact set of Hausdorff dimension at most one. We shall prove it has the universal differentiability property; it will follow, by Lemma 1.5, its Hausdorff dimension is equal to one.

Let $y \in S$, $\rho > 0$ and $g \colon \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function. We shall prove the existence of a point $x \in S$ of differentiability of g with $\|x - y\| < \rho$.

We may assume $\mathrm{Lip}(g) > 0$. Let H be the Hilbert space $\mathbb{R}^d$. We may pick $q < \lambda_1 := 1$ and $y' \in T_q$ with $\|y' - y\| < \rho/3$.

Let $\lambda_0 \in (q, 1)$. By applying Lemma 2.6 with $\eta < 1/2$ we can find distinct $r, s \in R \cap B_{\rho/3}(y')$ so that $[r, s] \subseteq T_{\lambda_0}$. Then, by Lebesgue's theorem, there exists $x_0 \in [r, s]$ such that $(x_0, e_0) \in D^g_{\lambda_0}$, where

$$e_0 = \frac{r - s}{\|r - s\|}.$$

Set $f_0 = g$, $K = 25\sqrt{2\mathrm{Lip}(g)}$, $\delta_0 = \rho/3$ and $\mu = \mathrm{Lip}(g)$.

Let the Lipschitz function f, the pair (x, e), $\lambda \in (\lambda_0, \lambda_1) = (\lambda_0, 1)$ and, for each $\varepsilon > 0$, the numbers $\delta_\varepsilon > 0$ and $\lambda_\varepsilon \in (\lambda, 1)$ be given by the conclusion of Theorem 2.7. We verify the conditions of Lemma 2.8 hold for the function $f \colon \mathbb{R}^d \to \mathbb{R}$, the pair $(x, e) \in D^f_\lambda$ and the family of sets $\{F_\varepsilon \subseteq \mathbb{R}^d \mid \varepsilon > 0\}$ where

$$F_\varepsilon = T_{\lambda_\varepsilon} \cap B_{\delta_\varepsilon}(x).$$

We know from Theorem 2.7 that the derivative $f'(x, e)$ exists and is non-negative. To verify condition (1) of Lemma 2.8, for every $\varepsilon > 0$ and $\eta \in (0, 1)$, we put $\psi_\varepsilon = \lambda_\varepsilon - \lambda$ and define

$$\delta_* = \frac{1}{2}\min\Big\{\Delta_0\Big(\frac{\eta}{2}, \psi_\varepsilon\Big), \delta_\varepsilon, 1 - \|x\|\Big\},$$

where Δ_0 is given by Lemma 2.6. We see that $\delta \in (0, \delta_*)$ implies

$$2\delta < \min\Big\{\Delta_0\Big(\frac{\eta}{2}, \psi_\varepsilon\Big), 1 - \|x\|\Big\}.$$

By Lemma 2.6 we can find $\alpha \in (0, \eta/2 \cdot 2\delta) = (0, \eta\delta)$ such that for every $r, s \in R(\alpha) \cap B_{2\delta}(x)$ we have $[r, s] \subseteq T_{\lambda_\varepsilon}$. Using the definition of $R(\alpha)$ and $B_{2\delta}(x) \subseteq B_1(0)$ we can find $x + \delta u_i' \in B_\alpha(x + \delta u_i)$ such that

$$[x + \delta u_1', x + \delta u_3'] \cup [x + \delta u_3', x + \delta u_2'] \subseteq T_{\lambda_\varepsilon}.$$

Note then that since

$$\|(x + \delta u_i') - (x + \delta u_i)\| < \alpha < \eta\delta,$$

we have $\|u_i' - u_i\| < \eta$; also as $\delta(1 + \eta) < 2\delta_* \le \delta_\varepsilon$ we have $x + \delta u_i' \in B_{\delta_\varepsilon}(x)$ for each $i = 1, 2, 3$. Thus

$$[x + \delta u_1', x + \delta u_3'] \cup [x + \delta u_3', x + \delta u_2'] \subseteq F_{\lambda_\varepsilon}.$$

Condition (2) of Lemma 2.8 is immediate from the definition of F_ε and equation (2.4) as $\mathrm{Lip}(f) \le \mathrm{Lip}(g) + \mu = 2\mathrm{Lip}(g)$ so that $25\sqrt{\mathrm{Lip}(f)} \le K$.

Therefore, by Lemma 2.8 the function f is differentiable at x. So too, therefore, is g as $(g - f)$ is linear. Finally, note that $x \in T_\lambda \subseteq S$ and

$$\|x - y\| \leq \|x - x_0\| + \|x_0 - y'\| + \|y' - y\| < \rho/3 + \rho/3 + \rho/3 = \rho. \qquad \square$$

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